# A Second-Order Approximation to Technology Choices

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**Abstract**

Resources used in scientific activities are, as any other, scarce. Hence, the economic system has, in every time moment, to choose how to allocate technological inputs. A technology choices model is developed, where scarce scientific resources are alternatively allocated to basic science activities and to applied technology uses. We find that saddle path stability holds for a not too high intertemporal discount rate. The accomplished result is found for a generic quadratic objective function, that is, for a second-order Taylor series approximation of a felicity function regarding technology development goals.

Keywords: Technology, Optimal control, Transitional dynamics, Saddle-path stability, Taylor-series expansion.

JEL classification: C61, O33

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## 1. Introduction

Economists have always attributed to technical change a prominent role in the way material wealth is generated. The endogenous growth literature aimed to explain the critical role of technology, either through the consideration of innovation externalities [as in Romer (1986) and Lucas (1988)] or by the explicit introduction of monopolistic competition technology production [Romer (1990), Grossman and Helpman (1991), aghion and Howitt (1992), Segerstrom, Anant and Dinopoulos (1990)].

After this initial effort relating technological activities modelling, many other studies have highlighted the role of technology, mainly in what concerns its role in the promotion of economic growth. The importance of technology and innovation to growth is exhaustively worked out in Aghion and Howitt (1998). Also Jones (1995), Stokey (1995), Galor and Tsiddon (1997), Acemoglu (2002), Aghion (2002) and Alcouffe and Kuhn (2004) constitute important references in what concerns the economic treatment and discussion of technology issues from a macro point of view.

For instance, Jones (1995) included technology in a growth model through the consideration of an R&D sector, where diminishing returns are present; given that knowledge is a non rival input, the same technology is used to generate more knowledge and to produce final goods, and in this way the decreasing returns effect can be eliminated and sustained growth will prevail. Furthermore, as Howitt (1999) and Jones (2003) argue, a link may be established between our world's evidence of increasing population growth and faster technological progress; Jones (2003) sustains that "More people means more Isaac Newtons and therefore more ideas. More ideas, because of nonrivalry, mean more per capita income. Therefore, population growth, combined with the increasing returns to scale associated with ideas, delivers sustained long run growth." (page 505).

A new impulse to the study of the economic impact of innovation is given by the changes in communication instruments – the information and communication technologies allow to highlight once again the way in which innovation is critical to the generation of wealth. Jaffe and Trajtenberg (2002) present a clear view of the technology new dimensions, "In the last few decades, we have experienced what have come to be called the 'information age' and the 'knowledge economy'. Hype aside, these labels do reflect a very real transformation: it is now 'knowledge' – not labor, machines, land or natural resources – that is the key economic asset that drives long-run economic performance." (page 1).

This paper intends to present an alternative view of technology generation issues. The non rival nature of technology and the industrial organization of innovation activities that lead unavoidably to a monopolistic competition market structure, are not our main concern. Instead, we assume a central planner framework, where a representative agent has to choose how to allocate scarce scientific resources. Our argument is that while the output of technological activities is, in part, non rival, the same is not true for a significant share of the inputs used in the generation of knowledge, in such a way that economic authorities have to give guidance in how to allocate such inputs in order to maximize welfare. For instance, the innovation process relies significantly on the human factor; scientists, however, cannot be in the same place at the same time and therefore they must be allocated to the tasks that best serve the purposes of wealth creation and, ultimately, welfare enhancing.

The present approach distinguishes between two types of knowledge: basic science and applied knowledge. This distinction is just the separation that generally is made between the 'R' and the 'D'; research activities serve the purpose of expanding what Acemoglu (2003) calls the 'Innovation possibilities frontier', and development implies using the available knowledge in directly productive activities. This applied variable may be thought has the technology index that can be found in a final goods aggregate production function.

Then, the economic system has a choice to make, which consists in deciding how to employ technology inputs. On one hand, if the innovation effort is mainly concerned with the knowledge frontier, less resources will be dedicated to the efficiency of the production process. On the other hand, trying to apply immediately all the existent knowledge to the generation of physical goods, scientific research is neglected and we may end with a state where no new knowledge is available to apply.

Our main task is to solve an optimal control problem where the previous trade-off is evidenced. We rely on the Nelson and Phelps (1966) analysis concerning basic-applied technology and we search for stability conditions that guarantee the existence of a system where a convergence process to the steady state point holds. This steady state point must be one in which a positive and constant rate of scientific progress is compatible with an optimal percentage of applied knowledge relatively to the reference frontier. The undertaken analysis is generic in the sense that we work with an objective function that obeys some ground rules but that does not display an explicit functional form.[1]

To maintain the analysis at a generic level, we use a procedure similar to the one that Woodford (2003) and Benigno and Woodford (2004) resort to for the analysis of monetary policy – the solution of our optimal control problem is characterized for initial conditions near the steady state and the assumed objective function will be approximated using a second-order Taylor expansion around the steady state.

The paper is organized as follows. Section 2 presents the model's main features, section 3 proceeds with the analysis of the model, considering a second-order approximation to the objective function, section 4 searches for conditions in which convergence to the steady state is feasible, and finally section 5 concludes.

## 2. The Technology Setup

Let $T(t)$ be the knowledge frontier level in moment $t$ and $A(t)$ the applied technology index. Given these two variables, the representative agent wants to maximize a function that has as arguments the growth rate of the technology level frontier and a technology gap variable that is the ratio between ready-to-use and theoretical knowledge. Hence, we define $\tau(t) \equiv \dot{T}(t)/T(t) - a(.)$ and $\phi(t) \equiv A(t)/T(t)$. In the first definition, variable $\tau(t)$ is the controllable part of technology growth; all the factors that promote growth and that are not controllable through the allocation of resources to each of the scientific activities are summarized in exogenous variable $a(.)$.

The objective function that we consider is $v[\phi(t), \tau(t)]$. We assume that $v$ is an increasing, concave, and smooth (infinitely many times differentiable) function. Thus, the following derivatives signs hold: $v_\phi>0$, $v_\tau>0$, $v_{\phi\tau}= v_{\tau\phi}>0$, $v_{\phi\phi}<0$, $v_{\tau\tau}<0$.

We consider a dynamic intertemporal model, what implies that the representative agent goal is to maximize the stream of $v$ functions from the present moment to some future horizon. We assume an infinite horizon; we also assume a constant discount rate, $\rho>0$. The problem is $\underset{\tau(t)}{Max}\int_0^{+\infty} v[\phi(t),\tau(t)].e^{-\rho.t}.dt$; note that $\tau(t)$ is the problem's control variable – the representative agent has the ability to control the pace of scientific progress, through the chosen allocation of technological resources. The technology indexes are state variables in the sense that there will be pre-defined motion rules that

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[1] In Gomes (2004), a Cobb-Douglas type objective function was considered, and in this case a saddle-path stable equilibrium is found, for reasonable bounds on parameter values.

determine their time evolution. The time evolution of *T*(*t*) is the one that is implicit in the definition of the science growth rate,

$$\dot{T}(t) = [a(.) + \tau(t)].T(t), T(0) = T_0 \text{ given} \tag{1}$$

In what concerns the differential equation governing the movement of *A*(*t*), we rely on Nelson and Phelps (1966) who present an expression similar to (2).

$$\dot{A}(t) = h(.).[T(t) - A(t)], h(.) > 0, A(0) = A_0 \text{ given} \tag{2}$$

Equation (2) indicates that the time evolution of *A*(*t*) is a function of two entities. First, a set of exogenous factors, represented by *h*(.); these can be for instance management skills or an environment that favours learning-by-doing. Second, a technology gap term; the economic interpretation of this technology gap term is as follows: the larger the distance between the truly available knowledge and what is currently used to productive uses, the stronger will be the tendency for knowledge to be applied to useful ends. The idea is one of a convergence process: it would be easier to find useful applications for scientific knowledge when a small fraction of such knowledge is in use than when a large portion of the economy's stock of ideas is already employed.

Recovering the technology ratio notion and combining (1) and (2), we find a unique state constraint for the optimal control problem,

$$\dot{\phi}(t) = h(.).[1 - \phi(t)] - [a(.) + \tau(t)].\phi(t), \ \phi(0) = A_0 / T_0 \tag{3}$$

Note that $0 \leq \phi(t) \leq 1$, because $A(t) \leq T(t)$.

Synthesizing, we have built a simple technology choices model. A representative agent has to choose the optimal science growth rate having in mind two conflicting objectives. One of these goals is precisely to amplify the knowledge frontier; the other one, consists in reducing the technology gap, that is, consists in putting to use the stock of ideas that the economy is able to accumulate. Expression (3) is the resource constraint that the economy faces regarding technology choices.

## 3. The Approximation of a General Objective Function

Our main goal is to derive a condition under which the problem defined in the previous section yields a stable steady state, that is, one wants to inquire how technology choices made under an optimal setup can lead to a stability outcome where independently of the initial state ($\phi_0$, $\tau_0$), the steady state $(\bar{\phi},\bar{\tau})$ is accomplished, as long as ($\phi_0$, $\tau_0$) is in the vicinity of $(\bar{\phi},\bar{\tau})$. Note what an unstable result would mean – one of two possibilities would prevail: $\bar{\phi}\rightarrow 0$ and $\bar{\tau}\rightarrow$ some upper bound, or $\bar{\phi}\rightarrow 1$ and $\bar{\tau}\rightarrow 0$. As it is easy to understand, none of the cases is economically meaningful – it does not make sense an impressive scientific progress if there is no application of such knowledge to practical uses; similarly, it is useless to have all knowledge applied to the generation of physical goods if there is no growth of technology possibilities. Therefore, the aim is to accomplish a steady state where a reasonable technology growth rate and a $\bar{\phi}\in(0,1)$ are observable. As it is obvious, the steady state is attained solely if stability holds. Then, a condition for stability has to be derived.

First, regard that the steady state point is the solution for the steady state static problem Max $v(\bar{\phi},\bar{\tau})$ subject to $\bar{\tau}=h(.).\dfrac{1-\bar{\phi}}{\bar{\phi}}-a(.)$. From this problem, we compute a second relation between $\bar{\phi}$ and $\bar{\tau}$, which is $\bar{\phi}^2=h(.).\dfrac{\bar{v}_\tau}{\bar{v}_\phi}$, where $\bar{v}_\phi$ and $\bar{v}_\tau$ are objective function's derivatives, evaluated in the steady state point. A system containing the two presented relations between $\bar{\phi}$ and $\bar{\tau}$ allows for solving for the determination of the steady state pair of values.

To find stability conditions, we consider the steady state point $(\bar{\phi},\bar{\tau})$ and compute a second-order Taylor series approximation to our objective function, expanding around the steady state result. The derivation yields,

$$v[\phi(t),\tau(t)]=$$
$$\alpha+\beta.\phi(t)+\gamma.\tau(t)+\bar{v}_{\phi\tau}.\phi(t).\tau(t)+\tfrac{1}{2}.\bar{v}_{\phi\phi}.\phi(t)^2+\tfrac{1}{2}.\bar{v}_{\tau\tau}.\tau(t)^2+O\left(\left\|\phi(t),\tau(t)\right\|^3\right) \tag{4}$$

where,

$$\alpha\equiv\bar{v}-\bar{v}_\phi.\bar{\phi}-\bar{v}_\tau.\bar{\tau}+\bar{v}_{\phi\tau}.\bar{\phi}.\bar{\tau}+\tfrac{1}{2}.\bar{v}_{\phi\phi}.\bar{\phi}^2+\tfrac{1}{2}.\bar{v}_{\tau\tau}.\bar{\tau}^2$$

$$\beta\equiv\bar{v}_\phi-\bar{v}_{\phi\phi}.\bar{\phi}-\bar{v}_{\phi\tau}.\bar{\tau}$$

$$\gamma \equiv \bar{v}_\tau - \bar{v}_{\phi\tau}.\bar{\phi} - \bar{v}_{\tau\tau}.\bar{\tau}$$

and $\bar{v}$ is the objective function value for steady state values of $\phi(t)$ and $\tau(t)$. Also, $\bar{v}_{\phi\tau}$, $\bar{v}_{\phi\phi}$ and $\bar{v}_{\tau\tau}$ represent second-order derivatives of the objective function, with variables replaced by the correspondent steady state values. The term $O\left(\left\|\phi(t),\tau(t)\right\|^3\right)$ is a residual term, relating higher-order derivatives which, for simplifying purposes, are not computed.

Solving the technology choices problem under the approximated $v$ function in (4) we will find an expression for the time evolution of the technology growth rate variable. To do this, we use Pontryagin's principle and calculate first order optimality conditions.

Assume a co-state variable $p(t)$ and consider the following current-value Hamiltonian function,

$$\aleph\left[\phi(t),\tau(t),p(t)\right] \equiv v\left[\phi(t),\tau(t)\right] + p(t).\left\{h(.).\left[1-\phi(t)\right] - \left[a(.)+\tau(t)\right].\phi(t)\right\} \tag{5}$$

The necessary optimality conditions are,

$$\aleph_\tau = 0 \Rightarrow \gamma + \bar{v}_{\phi\tau}.\phi(t) + \bar{v}_{\tau\tau}.\tau(t) = p(t).\phi(t) \tag{6}$$

$$\aleph_\phi = \rho.p(t) - \dot{p}(t) \Rightarrow \dot{p}(t) = \left[\rho + h(.) + a(.) + \tau(t)\right].p(t) - \beta - \bar{v}_{\phi\tau}.\tau(t) - \bar{v}_{\phi\phi}.\phi(t) \tag{7}$$

The transversality condition $\lim_{t\to+\infty} \phi(t).e^{-\rho.t}.p(t) = 0$ also applies.

Differentiating (6) in order to time, one gets the expression,

$$\dot{p}(t) = \frac{1}{\phi(t)}.\left[\bar{v}_{\tau\tau}.\dot{\tau}(t) + (\bar{v}_{\phi\tau} - p(t)).\dot{\phi}(t)\right] \tag{8}$$

Replacing (3) and (7) in (8), and rearranging, the following is the dynamic equation that reflects how the growth rate of technology evolves in time,

$$\dot{\tau}(t)=\frac{1}{\bar{v}_{\tau\tau}}.\{[\rho+h(.)+a(.)+\tau(t)].[\gamma+\bar{v}_{\phi\tau}.\phi(t)+\bar{v}_{\tau\tau}.\tau(t)]- \\ -\beta.\phi(t)-\bar{v}_{\phi\tau}.\phi(t).\tau(t)-\bar{v}_{\phi\phi}.\phi(t)^2+[\gamma+\bar{v}_{\tau\tau}.\tau(t)].[h(.).(1/\phi(t)-1)-a(.)-\tau(t)]\} \tag{9}$$

## 4. Conditions for Saddle-Path Stability

Although (9) is a somehow heavy expression, equations (3) and (9) allow for the determination of a linearized system in the steady state vicinity from which it is possible to characterize stability results. The linearized system is

$$\begin{bmatrix}\dot{\phi}(t)\\ \dot{\tau}(t)\end{bmatrix}=J.\begin{bmatrix}\phi(t)-\bar{\phi}\\ \tau(t)-\bar{\tau}\end{bmatrix},\quad J=\begin{bmatrix}-[h(.)+a(.)+\bar{\tau}] & -\bar{\phi}\\ \omega & \rho+h(.)+a(.)+\bar{\tau}\end{bmatrix} \tag{10}$$

with $\omega\equiv\frac{1}{\bar{v}_{\tau\tau}}.\left[\rho.\bar{v}_{\phi\tau}-\bar{v}_{\phi}-\bar{v}_{\phi\phi}.\bar{\phi}-\bar{v}_{\tau}.\frac{h(.)}{\bar{\phi}^2}\right]$.

Noticing that $Tr(J)=\rho$ and $Det(J)=\omega.\bar{\phi}-[h(.)+a(.)+\bar{\tau}].[\rho+h(.)+a(.)+\bar{\tau}]$, one is able to state the stability condition. The trace value indicates that the system is not globally stable ($J$ cannot have two negative eigenvalues). To guarantee saddle-path stability, it is necessary that $Det(J)<0$. To simplify the determinant expression note that $h(.)+a(.)+\bar{\tau}=\frac{h(.)}{\bar{\phi}}$ and $\rho+h(.)+a(.)+\bar{\tau}=\frac{\bar{v}_{\phi}}{\bar{v}_{\tau}}.\bar{\phi}$ [this last relation is attainable through (9)]. Hence, saddle-path stability implies the condition $\omega<\frac{\bar{v}_{\phi}}{\bar{v}_{\tau}}.\frac{h(.)}{\bar{\phi}}$. Rearranging in order to isolate the discount rate, we get

$$\rho<\frac{1}{\bar{v}_{\phi\tau}}.\left\{\bar{v}_{\phi}+\bar{v}_{\phi\phi}.\bar{\phi}+\frac{\bar{v}_{\phi}}{\bar{v}_{\tau}}.\bar{v}_{\tau\tau}.\frac{h(.)}{\bar{\phi}}+\bar{v}_{\tau}.\frac{h(.)}{\bar{\phi}^2}\right\} \tag{11}$$

Inequality (11) is a necessary condition for saddle-path stability. Remember that $\bar{v}_{\phi\phi}<0$, $\bar{v}_{\tau\tau}<0$, and that the other objective function derivatives are positive functions; thus, the right-hand side of expression (11) may be a positive or a negative value. If it is a negative value, it does not make sense to assume that saddle-path stability holds, because a negative discount rate is counter-intuitive given our model's assumptions. For a positive right-hand side value, a saddle-path equilibrium is accomplished for small discounting of future technological achievements.

## 5. Final Remarks

We have identified two important goals relating technology choices in society: the expansion of knowledge frontiers and the use of ideas to generate economic value. Assuming that scientific and technological inputs are (partially) rival inputs, it is not possible, under optimality conditions, to promote one of the objectives without injuring the other. Thus, one needs to identify the paths for the technology growth rate and for the applied-basic technology ratio that maximize an objective function that gives attention to both goals.

The model is solved for an objective function general form that is approximated in the steady state vicinity using a second-order Taylor series expansion. Regarding that instability leads to an undesirable solution of no technical progress serving the purpose of rising inputs efficiency, we search for the condition for stability under our second-order approximated model. One finds that full stability is out of the question, but a saddle-path result is possible. The condition that guarantees the existence of a dimension one stable trajectory through which the variables may converge to the steady state is such that imposes a upper bound on the discount rate at which the representative agent values future technological achievements.

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